Math 1522 - Exam 3 Study Guide

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Contents

Summary and Disclaimer

This is a study guide for the first exam for math 1522 at the University of New Mexico (Calculus II). The exam covers sections 11.1 - 11.6 of Stewart's Calculus. As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material from a calculus 1 course, including solving integrals with u-substitution. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

If you are not in my class, I cannot guarantee how much these notes will help you. With that said, if your TA or instructor has shared these with you, then you will most likely get some use out of them.

Methods and Techniques

Our first concern is with evaluating the limits of sequences. Here, the following method is quite useful:

Limits of Sequences

Consider a sequence $\{a_n\}$ where $a_n = f(n)$ for some function $f(x)$. If

$$\lim_{x\to\infty} f(x)$$

exists, then it is equal to the limit of the sequence. That is,

$$\lim_{x\to\infty} f(x) = \lim_{n\to\infty} f(n).$$

When we can't evaluate the above limit, the best way to find the limit of the sequence is to just start listing terms. We will do an example of this later.

The next important subject for the exam is geometric series.

Geometric Series Formula

The if a geometric series has $|r| < 1$, then it has the following value

$$\sum_{n=k}^{\infty} a_n = \frac{a_k}{1-r}$$

where r is the ratio of the series.

The above formula lets you start at any value k. This means that you could solve sums starting at 1 and sums starting at 1000 just as easily with this formula!

Next up, we have telescoping sums.

Telescoping Sums

These are sums of the form
$$\sum_{n=1}^{\infty} f(n) - f(n+k)$$
where k is some positive number. The way to solve these is to write out the actual values of the partial sums
$$s_N = f(1) - f(1+k) + f(2) - f(2+k) + \ldots + f(N) - f(N+k).$$
And then evaluate the limit
$$\lim_{N\to\infty} s_N.$$

This is best learned with examples, which we will see some of later.

The next item on our list is the divergence test. This tells us about when a series needs to be divergent. There are divergent series that do not pass this test, so be careful when interpreting the results!

Divergence Test

Consider the series
$$\sum_{n=1}^{\infty} a_n.$$
If
$$\lim_{n\to\infty} a_n \neq 0,$$
then the series must diverge. If the limit goes to zero, then the test tells you nothing about the series.

It is worth noting that if the limit is undefined (or even diverges), then it is certainly not zero! So the above test still applies in these situations.

Our next test is the integral test:

Integral Test

If $f(x)$ is continuous and eventually decreasing, then
$$\sum_{n=k}^{\infty} f(n)$$
converges if

$$\int_k^\infty f(x)\ dx$$

is converges, and diverges otherwise.

This, combined with a previous homework exercise, leads us right into the next test:

p-test

The series

$$\sum_{n=k}^\infty \frac{1}{n^p}$$

converges if $p > 1$ and diverges otherwise.

This test is often not used on its own, but rather is used in conjunction with one of the two comparison tests:

Comparison Test

If $a_n \leq b_n$, then if

$$\sum_{n=k}^\infty b_n$$

converges, so does

$$\sum_{n=k}^\infty a_n.$$

On the other hand, if

$$\sum_{n=k}^\infty a_n$$

diverges, then so does

$$\sum_{n=k}^\infty b_n.$$

Limit Comparison Test

Consider a_n and b_n as two non-negative sequences. Then if

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

is a positive (non-zero) finite number, then

$$\sum_{n=k}^{\infty} a_n$$

and

$$\sum_{n=k}^{\infty} b_n$$

either both converge or both diverge.

We typically use the first of the two comparison tests when we know what we can easily compare to. We use the second of the two comparison tests in the other situations, since the limits are often easy to compute, and we can compare to a p-series.

Next, we have the ratio test, which works when the sequence is non-zero:

Ratio Test

Let a_n be given. If

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

is defined, then if $L > 1$, the series diverges, if $L < 1$ the series converges absolutely, and if $L = 1$ the test is inconclusive.

We can apply this to alternating series test as a consequence:

Alternating Series Test

If $\lim_{n\to\infty} b_n = 0$ and b_n is a decreasing sequence, then

$$\sum_{n=k}^{\infty} (-1)^n b_n$$

converges.

At this point, it is also worth point mentioning two things about alternating series. The first is the definition of absolute convergence:

Absolute Convergence

The series

$$\sum_{n=k}^{\infty} a_n$$

is said to converge absolutely if

$$\sum_{n=k}^{\infty} |a_n|$$

converges.

Finally, we have the error on alternating series:

Error on Alternating Series

The difference between the nth partial sum and the actual value of

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

is less than b_{n+1}.

Worked Examples

Example: Find the value of
$$\lim_{n\to\infty} \frac{n^2+1}{n^2-1}.$$
To do this, we will compute the regular limit:
$$\lim_{x\to\infty} \frac{x^2+1}{x^2-1} = 1.$$
This follows from applying L'Hopital's rule twice. So,
$$\lim_{n\to\infty} \frac{n^2+1}{n^2-1} = 1$$
as well.

Example: Find the value of
$$\lim_{n\to\infty} \sin(\pi n).$$
Here, we will compute the value of each term explicitly, since the regular limit will not converge. Our sequence is:
$$\sin(\pi), \sin(2\pi), \sin(3\pi), \sin(4\pi), \ldots.$$
After evaluating each of these, we get
$$0, 0, 0, \ldots.$$
So,
$$\lim_{n\to\infty} \sin(\pi n) = 0.$$

Example: Determine if the series converges absolutely, converges dependently, or diverges.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
This series converges dependently, but not absolutely. To see that it doesn't converge absolutely, note that
$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{|(-1)^n|}{|n|} = \sum_{n=1}^{\infty} \frac{1}{n}$$
which diverges.
To see that this converges dependently, we will use the alternating series test. Here,

since a_n is of the form $(-1)^n b_n$ (where $b_n = \frac{1}{n}$), we need to compute the relevant limit:

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

so the first part of the alternating series test is satisfied. Moreover, $\frac{1}{n}$ is decreasing, so the second part of the test is also satisfied. Thus, the alternating series converges.

Example: For what values of x does the series

$$\sum_{n=1}^{\infty} \left(x - \frac{1}{2}\right)^n$$

converge? What is the value at these points?
To solve this, we note that this is a geometric series with $r = x - \frac{1}{2}$. So, it converges when $\left|x - \frac{1}{2}\right| < 1$. But this means that $-1 < x - \frac{1}{2} < 1$, or $-\frac{1}{2} < x < \frac{3}{2}$.
To find what value the series converges to at these points, we note that the terms of the series are

$$x - \frac{1}{2}, \left(x - \frac{1}{2}\right)^2, \left(x - \frac{1}{2}\right)^3, \ldots.$$

So, by our formula it converges to

$$\frac{x - \frac{1}{2}}{1 - \left(x - \frac{1}{2}\right)} = \frac{x - \frac{1}{2}}{\frac{3}{2} - x}.$$

If one desired, this may be simplified further to

$$\frac{2x - 1}{3 - 2x}.$$

Practice Problems

1. How many terms of the series
$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^n}$$
do you need to add so that the error in approximating the series is less than 3^{-3}?

2. Determine if the series converges absolutely, conditionally, or diverges:
$$\sum_{n=3}^{\infty}(-1)^n\frac{1}{\sqrt{n}}.$$

3. Determine if the series converges or diverges, and find the value if possible:
$$\sum_{n=1}^{\infty}2^n 3^{-n+2}$$

4. Determine if the series converges or diverges, and find the value if possible:
$$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$$

5. For what values of x does the following series converge? What value does it converge to when it converges?
$$\sum_{n=1}^{\infty}\left(\frac{x}{2}+1\right)^n$$

Practice Problem Solutions

1.

Solution: For this problem, we want to use our formula for the error on alternating series. Since here $b_n = \frac{1}{3^n}$, and we want to find when this is less than $\frac{1}{3^{-3}}$. Since this is the third term of the positive part of our series, we have that

$$\left|\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^n} - \sum_{n=0}^{2}(-1)^n\frac{1}{3^n}\right| < \frac{1}{3^3} = 3^{-3}.$$

So, we need to find out how many terms are in the sum

$$\sum_{n=0}^{2}(-1)^n\frac{1}{3^n},$$

which is three terms, since

$$\sum_{n=0}^{2}(-1)^n\frac{1}{3^n} = 1 - \frac{1}{3} + \frac{1}{9}.$$

2.

Solution: The series converges by the alternating series test, since

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$$

and $\frac{1}{\sqrt{n}}$ is decreasing. However, it does not converge absolutely, since

$$\sum_{n=3}^{\infty}\left|(-1)^n\frac{1}{\sqrt{n}}\right| = \sum_{n=3}^{\infty}\frac{1}{|\sqrt{n}|} = \sum_{n=3}^{\infty}\frac{1}{\sqrt{n}}$$

which diverges by the p-test.

3.

Solution: We note that this series is geometric. To see the pattern, we will write out the first few term of the sequence:

$$6, 4, \frac{8}{3}, \frac{16}{9}, \ldots.$$

So, $a = 6$ and $r = \frac{2}{3}$. So, by our formula we have that

$$\sum_{n=1}^{\infty} 2^n 3^{-n+2} = \frac{6}{1-\frac{2}{3}} = 6 \cdot \frac{3}{1} = 18.$$

4.

Solution: By the p-test, this series converges, since $\frac{3}{2} > 1$. However, since this is not a geometric series and not a telescoping sequence, we cannot find a value for this series.

5.

Solution: This is a geometric series. We note that $r = \frac{x}{2} + 1$, and we write out a few terms of the sequence

$$\frac{x}{2} + 1, \left(\frac{x}{2} + 1\right)^2, \left(\frac{x}{2} + 1\right)^3, \ldots.$$

So, the first term is $a = \frac{x}{2} + 1$. So, the formula for our series is

$$\frac{\frac{x}{2} + 1}{1 - \left(\frac{x}{2} + 1\right)} = \frac{\frac{x}{2} + 1}{-\frac{x}{2}} = -\frac{x+2}{x}$$